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The symplectic structure of Euler–Lagrange superequations and Batalin–Vilkoviski formalism

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Abstract

We study the graded Euler–Lagrange equations from the viewpoint of graded Poincaré–Cartan forms. An application to a certain class of solutions of the Batalin–Vilkoviski master equation is also given.

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1. Introduction

It is well known that given a classical manifold M, which represents the configuration space of a physical system, the canonical symplectic structure of the cotangent bundle allows one to develop the Hamiltonian dynamics of that system in a very concise way. A similar approach can be followed in the case of the Lagrangian theory, which can be formulated on the tangent bundle of the configuration manifold constructing on it a symplectic form which, however, is not canonical and depends directly on the Lagrangian chosen to describe the system.

This classical theory of dynamical systems can be generalized in many ways. One of them includes anticommuting variables, which are suitable to describe quantum effects such as spin; in this direction, there has been previous work studying graded (or 'super') Hamiltonian mechanics, calculus of variations, etc (see, for example, [Mon 92, Ibo-Mar 93, Car-Fig 97, Mon-Muñ 02 or the recent book [Del 99]). The 'golden rule' of generalizations to the graded case establishes that commuting and anticommuting variables must be treated in exactly the same way, but in the majority of the literature we find an asymmetry when dealing with the evolution parameter, which is always taken to be the 'bosonic' time t, with no 'fermionic' counterpart. This is stressed in [Mon 92], where a graded Hamiltonian theory is developed taking into account the supermanifold $\mathbb{R}^{1|1}$ as the space for the evolution parameters (s, t) with s being an anticommuting variable. The reasons for this choice are not only aesthetic, but have strong mathematical support: as proved in [Mon-Sán 93], this is necessary if we want to integrate arbitrary graded vector fields, that is, to describe arbitrary graded systems (see also [Sha 80]).

Thus, our translation of the aforementioned 'golden rule', is the replacement of the classical parameter space $\mathbb{R} \equiv \mathbb{R}^{1|0}$ by $\mathbb{R}^{1|1}$. This procedure is not new in physics, where the use of $\mathbb{R}^{1|1}$ goes under the name '(1, 1)-superspace'. Even the most simple models based on it have found interesting applications: as early as in 1982, Witten (see [Wit 82]) suggested that the index formula could be understood in terms of a suitable quantum mechanical supersymmetric system, in which the parameter space is given by a pair (t, θ) with t being bosonic and θ fermionic (our (t, s)). His insight has been developed ever since in a lot of papers, among them we could cite [Alv 83, Fri-Win 84]. In the book by Freed [Fre 99] it is used to construct a quantization model for the superparticle (a detailed account can be consulted in [Del 99], pp 478 and ff), and also a justification for the need for the introduction of $\mathbb{R}^{1|1}$ is given (see pp 40–41 in [Fre 99]).

In accordance with these ideas, we develop a graded Lagrangian theory (which can be viewed as complementary to the Hamiltonian one of [Mon-Muñ 02]) in the following steps: starting from the graded bundle $\mathbb{R}^{1|1} \times (M, \mathcal{A}) \xrightarrow{\pi} \mathbb{R}^{1|1}$, and the corresponding bundle of graded 1-sections, denoted by $J^1_G(\pi)$, we take a graded Lagrangian L defined on it (thus depending on coordinates $\{t, s, x^a, x^a_t, x^a_s\}$), construct a certain graded 1-form associated with L (the Poincaré–Cartan 1-form) and then pass to $J^2_G(\pi)$ with the use of $\mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}$, the graded Lie derivative with respect to the horizontal lift of $\frac{\partial}{\partial s}$. The Poincaré–Cartan 1-form will allow us to construct a graded symplectic 2-form which projects onto a suitable sub-bundle, the space of solutions (with coordinates $\{t, s, x^a, x^a_t, x^a_s, x^a_{ts}\}$). The resulting form, Ω_L, is such that the integral curves of its graded Hamiltonian vector fields give the solutions to the Euler–Lagrange superequations.

As an application, we give a method for constructing a class of solutions of the master equation of the Batalin–Vilkoviski formalism in the context of particle supermechanics. There has been previous work in this direction, mainly by Däyi (see [Day 88, Day 93]), who uses an 'odd time' formulation. His results are based (in our notation) upon $\mathbb{R}^{0|1}$, but we insist on the naturality of the $\mathbb{R}^{1|1}$ formalism.

Throughout this paper, we will freely interchange the prefixes 'super-' and 'graded'.

2. Preliminaries

For generalities on graded manifolds, see [Kos 77].

2.1. Graded manifolds and curves on them

On a graded manifold $(M, \mathcal{A})$, of graded dimension (m, n) and structural sheaf of graded commutative algebras $\mathcal{A}$, positive indices are used for even coordinates: $x_i, i = 1, \ldots, m$, and negative indices for odd coordinates: $x_i, i = -n, \ldots, -1$. The index is just a label, and for ease of writing we will place it as a subscript or a superscript. The natural homomorphism is denoted by $\mathcal{A} \to C^\infty_M$, $f \mapsto \tilde{f}$. Nevertheless, the coordinates of the graded manifold $\mathbb{R}^{1|1}$—with base manifold $\mathbb{R}$ and graded ring $\mathcal{R}^{1|1}$—are denoted by $=(t, s)$, with $|t| = 0$, $|s| = 1$; i.e, $\mathcal{R}^{1|1} = \{f(t) + g(t)s : f, g \in C^\infty(\mathbb{R})\}$. In the category of graded manifolds, $\mathbb{R}^{1|1}$ plays the same role as $\mathbb{R}$ in the category of differentiable manifolds.

Recall that a homogeneous endomorphism $D \in End(\mathcal{A})$ is a graded derivation if it verifies the graded Leibniz rule:

$$D(fg) = D(f)g + (-1)^{|D||f|} f D(g)$$

where $|D|$ is its degree, that is, $D(A_\alpha) \subset A_{\alpha+|D|}$. A fundamental result is that the space of such derivations, $Der(\mathcal{A})$, has the structure of graded Lie algebra when endowed with the restriction of the usual bracket of endomorphisms $[.,.]$: $End(\mathcal{A}) \times End(\mathcal{A}) \to End(\mathcal{A})$, given by

$$[F, G] = F \circ G - (-1)^{|F||G|} G \circ F. \tag{2.1}$$

The elements of $Der(\mathcal{A})$ are the graded vector fields of $(M, \mathcal{A})$ (just as $\mathcal{X}(N) = Der(C^\infty(N))$ in a usual manifold N). As in the non-graded case, the dual gives the graded differential forms:

$$\Omega_G(M) = \sum_{p\in\mathbb{N}} \Omega_G^p(M) \qquad \text{where} \quad \Omega_G^p(M) = \Lambda^p(Der^*(\mathcal{A})).$$

Each $\Omega_G^p(M)$ itself is a graded $\mathcal{A}$-module, and several morphisms on $\Omega_G(M)$ can be defined as usual (see [Kos 77]): the graded exterior differential d^G, the insertion of a $D \in Der(\mathcal{A})$ ι_D, etc. On the bigraded $\mathcal{A}$-module $End_{\mathcal{A}}(\Omega_G(M))$, we can introduce a commutator analogous to (2.1), now taking into account the bigrading: if F has bidegree (f_1, f_2) and G has bidegree (g_1, g_2), we define

$$[F, G] = F \circ G - (-1)^{f_1 g_1 + f_2 g_2} G \circ F$$

and this is a new element of $End_{\mathcal{A}}(\Omega_G(M))$ with bidegree $(f_1 + g_1, f_2 + g_2)$. If $D \in Der(\mathcal{A})$, then the graded Lie derivative $\mathcal{L}_D^G$ is defined by

$$\mathcal{L}_D^G = [\iota_D, \mathrm{d}^G] = \iota_D \circ \mathrm{d}^G + \mathrm{d}^G \circ \iota_D.$$

We recall that a classical curve $\gamma : \mathbb{R} \to M$ can be seen as a section of $p_1 : \mathbb{R} \times M \to \mathbb{R}$. In the graded case we must substitute $\mathbb{R}^{1|1}$ for $\mathbb{R}$. Hence a graded curve must be a section of the graded submersion $p_1 : \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$ given by the projection onto the first factor, or equivalently, a morphism of graded manifolds $\gamma : \mathbb{R}^{1|1} \to (M, \mathcal{A})$.

Example 1. In order to work out an example, let us choose a particular graded manifold. Let M be a differentiable manifold and let us consider the graded manifold (M, Ω_M), where Ω_M denotes the sheaf of differential forms on M. Hence $\dim(M, \Omega_M) = (m, m)$ if $m = \dim M$. Given a coordinate system $\{y_i\}_{i=1}^m$ on M, we can build up a system of adapted graded coordinates: $\{y_i, \mathrm{d}y_i\}_{i=1}^m$. According to our way of denoting graded coordinates, $\{x_i\}$, $i = -m, \ldots, -1, 1, \ldots, m$, we have $x_i = y_i$, $x_{-i} = \mathrm{d}y_i$, $i = 1, \ldots, m$. A graded curve $\gamma : \mathbb{R}^{1|1} \to (M, \Omega_M)$ is determined by a pair of maps $\gamma : \mathbb{R} \to M$, $\gamma^* : \Omega(M) \to \mathcal{R}^{1|1}$. Note that the homomorphism γ^* is not necessarily the pull-back map of $\gamma : \mathbb{R} \to M$. We have

$$\begin{cases} \gamma^*(y_i) = y_i \circ \gamma = f^i(t) \\ \gamma^*(\mathrm{d}y_i) = g^i(t)s \end{cases} \qquad f^i, g^i \in C^\infty(\mathbb{R}) \qquad i = 1, \ldots, m. \tag{2.2}$$

If γ^* is the pull-back of $\gamma : \mathbb{R} \to M$, then $g^i = (f^i)'$.

2.2. *Graded jet bundles*

The usual coordinate description of jet bundles does not work with graded manifolds. A more intrinsic construction of graded jet bundles, entailing further algebraic formalizations, is needed. This is done in [Her-Mun 84] and we do not repeat it here. According to this we will briefly recall the construction for the particular case of the graded 1-jet bundle $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ of local sections of $p_1 : \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$; its graded dimension is $(1 + 2m + n, 1 + m + 2n)$, where $(m, n) = \dim(M, \mathcal{A})$. We remark that the base manifold of $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ is not equal to $J^1(\mathbb{R}, M)$. The graded ring of $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ is denoted by $\mathcal{A}^1$. Also, we denote the graded ring of $\mathbb{R}^{1|1} \times (M, \mathcal{A})$ by $\mathcal{A}^0$ as $J^0(\mathbb{R}^{1|1}, (M, \mathcal{A})) = \mathbb{R}^{1|1} \times (M, \mathcal{A})$.

The graded fibred coordinates on first-order jet bundles are defined in [Her-Mun 84, Her-Mun 85, Mon 92] and denoted by $\{t, s, x_i, x_t^i, x_s^i\}$, $i = -n, \ldots, -1, 1, \ldots, m$: the even coordinates are t; $\{x_i\}_{i=1}^m$; $\{x_t^i\}_{i=1}^m$; $\{x_s^i\}_{i=-n}^{-1}$, and the odd ones are s; $\{x_i\}_{i=-n}^{-1}$; $\{x_t^i\}_{i=-n}^{-1}$; $\{x_s^i\}_{i=1}^m$.

2.3. Curves and the first-order jet bundle

Let us recall that a variation of a classical curve $\gamma : \mathbb{R} \to M$ is a one-parameter family of curves $\bar{\gamma}_{\bar{t}}(t)$ ($\bar{t} \in \mathbb{R}$ being the variational parameter) such that $\bar{\gamma}_0 = \gamma$. According to our philosophy of substituting $\mathbb{R}^{1|1}$ for $\mathbb{R}$, a variation of a graded curve on a graded manifold, $\gamma : \mathbb{R}^{1|1} \to (M, \mathcal{A})$, must also be an $\mathbb{R}^{1|1}$-parameter family of graded curves.

We exclusively consider variations of a curve induced by a graded vector field. In the classical case, the variations of a curve induced by a vector field are just the composition of the curve with the integral flow of the vector field. It can be shown that any even graded vector field can be integrated by simply using an even parameter, but the situation is different in the odd case.

Let us briefly recall the problem of existence and uniqueness of solutions of first-order superdifferential equations that have been studied in [Mon-Sán 93]. The first fact to note is that the parameter space in the graded setting is $\mathbb{R}^{1|1}$ and that the problem of finding the integral flow of a graded vector field must be stated in terms of this parameter space. Second, once we have chosen the parameter space, we must choose a model of graded vector field on it. It is easy to check that there are three possible graded Lie algebra structures on $\mathbb{R}^{1|1}$ each giving rise to a different model of right-invariant graded vector field. For example, for the additive structure (the one we use in what follows) the model of graded vector field is given by $\partial/\partial t + \partial/\partial s$.

Let X be a graded vector field on the graded manifold $(M, \mathcal{A})$. We say that $\Gamma : \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to (M, \mathcal{A})$ is the flow of X if together with an initial condition the following equation holds:

$$ev_{t=0} \circ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) \circ \Gamma^* = ev_{t=0} \circ \Gamma^* \circ X$$

where $ev_{t=0}$ is the map defined by $=ev_{t=0}(f(t) + g(t)s) = f(0)$. In [Mon-Sán 93] it is shown that any graded vector field can be integrated, in the previous sense, by means of integral curves parametrized on $\mathbb{R}^{1|1}$. It is also shown there that if the homogeneous parts X_0, X_1 of X satisfy the equations $[X_0, X_1] = [X_1, X_1] = 0$, then the previous equation also holds without the evaluation map, i.e.,

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) \circ \Gamma^* = \Gamma^* \circ X. \tag{2.3}$$

Moreover the flow induces an action of the additive Lie group structure of $\mathbb{R}^{1|1}$ on the supermanifold $(M, \mathcal{A})$ and then a kind of relation like $\Gamma_{t_1}, s_1 \circ \Gamma_{t_2,s_2} = \Gamma_{t_1+t_2,s_1+s_2}$ is valid.

Example 2 (cf [Mon-Sán 93])**.** On the graded manifold (M, Ω_M) the exterior derivative d is an example of an odd vector field. Its integral flow is given by the map $\Gamma = (\Gamma, \Gamma^*) : \mathbb{R}^{1|1} \times (M, \Omega_M) \to (M, \Omega_M)$, with $\Gamma = \pi_M : \mathbb{R} \times M \to M$ and $\Gamma^* : \Omega(M) \to \mathcal{A}_{\mathbb{R}^{1|1} \times (M, \Omega_M)}$

is given by $\Gamma^*(\alpha) = \alpha + \bar{s}\,\mathrm{d}\alpha$, where $= \bar{t}, \bar{s}$ are the graded coordinates in $\mathbb{R}^{1|1}$. Therefore the variation of a graded curve (2.2), produced by the graded vector field d, is given by

$$\begin{aligned} \gamma^* \circ \Gamma^*(y_i) &= \gamma^*(y_i + \bar{s}\,\mathrm{d}y_i) = f^i(t) + g^i(t)\bar{s}s \\ \gamma^* \circ \Gamma^*(\mathrm{d}y_i) &= \gamma^*(\mathrm{d}y_i) = g^i(t)s. \end{aligned} \qquad i = 1, \ldots, m. \tag{2.4}$$

Example 3. Let us describe the 1-jet prolongation of a graded curve γ on (M, Ω_M). If γ^* is given by (2.2), then $j^1\gamma$ is determined by the following equations:

$$\begin{cases} j^1(\gamma)^*\big(x_t^i\big) = \frac{\partial}{\partial t}(f^i(t)) = (f^i)'(t) \\ j^1(\gamma)^*\big(x_s^i\big) = \frac{\partial}{\partial s}(\gamma^*(y_i)) = 0 \\ j^1(\gamma)^*\big(x_t^{-i}\big) = \frac{\partial}{\partial t}(g^i(t)s) = (g^i)'(t)s \\ j^1(\gamma)^*\big(x_s^{-i}\big) = \frac{\partial}{\partial s}(g^i(t)s) = g^i(t). \end{cases} \tag{2.5}$$

Moreover, for the curves (2.4), given by the variations produced by the graded vector field d, we have

$$\begin{cases} j^1(\gamma_{\bar{t},\bar{s}})^*\big(x_t^i\big) = \frac{\partial}{\partial t}(f^i(t) + g^i(t)\bar{s}s) = (f^i)'(t) + (g^i(t))' \\ j^1(\gamma_{\bar{t},\bar{s}})^*\big(x_s^i\big) = \frac{\partial}{\partial s}(\gamma^*_{\bar{t},\bar{s}}(y_i)) = \frac{\partial}{\partial s}(f^i(t) + g^i(t)\bar{s}s) = -g^i(t)\bar{s} \\ j^1(\gamma_{\bar{t},\bar{s}})^*\big(x_t^{-i}\big) = \frac{\partial}{\partial t}(g^i(t)s) = (g^i)'(t)s \\ j^1(\gamma_{\bar{t},\bar{s}})^*\big(x_s^{-i}\big) = \frac{\partial}{\partial s}(g^i(t)s) = g^i(t). \end{cases} \tag{2.6}$$

3. Graded Poincaré–Cartan forms

Let us begin by considering the Poincaré–Cartan 1-form associated with L, which can be written locally in a way analogous to that of the classical case, where we would put something like $(\mathrm{d}x - \dot{x}\,\mathrm{d}t)\frac{\partial L}{\partial \dot{x}} + L\,\mathrm{d}t$:

$$\Theta_0^L := \big(\mathrm{d}^G x^a - \mathrm{d}^G t \cdot x_t^a - \mathrm{d}^G s \cdot x_s^a\big)\frac{\partial L}{\partial x_t^a} + \mathrm{d}^G t \;\cdot\; L. \tag{3.1}$$

It is from this 1-form that we want to define our definitive form. But before doing this, in order to get a better understanding of the structure of Θ_0^L, let us rewrite it in terms of L and a canonical graded endomorphism of $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$, J:

$$J := \tilde{J} - \mathrm{d}^G t \otimes \Delta_1 - \mathrm{d}^G s \otimes \Delta_2$$

where

$$\tilde{J} = \mathrm{d}^G x^a \otimes \frac{\partial}{\partial x_t^a} \qquad \Delta_1 = x_t^a \frac{\partial L}{\partial x_t^a} \qquad \Delta_2 = x_s^a \frac{\partial L}{\partial x_t^a}.$$

We have then

$$\begin{aligned} \mathcal{L}_J^G L &= [i_J, \mathrm{d}^G](L) = i_J\,\mathrm{d}^G L \\ &= \Big(\mathrm{d}^G x^a \otimes i_{\frac{\partial}{\partial x_t^a}} - \mathrm{d}^G t \cdot x_t^a \otimes i_{\frac{\partial}{\partial x_t^a}} - \mathrm{d}^G s \cdot x_s^a \otimes i_{\frac{\partial}{\partial x_t^a}}\Big)\,\mathrm{d}^G L \\ &= \big(\mathrm{d}^G x^a - \mathrm{d}^G t \cdot x_t^a - \mathrm{d}^G s \cdot x_s^a\big)\frac{\partial L}{\partial x_t^a} \\ &= \Theta_0^L - \mathrm{d}^G t \cdot L \end{aligned}$$

that is,

$$\Theta_0^L = \mathcal{L}_J^G L + \mathrm{d}^G t \cdot L.$$

Note that this expression transfers the question about whether Θ_0^L is canonical to the same question but this time about J. (An intrinsic construction of J, which is rather technical, will be considered elsewhere [MMV 02]. For the moment, let us mention that J is the graded analogue of the $(1, m)$-tensor field S_η that appears in [Sau 89], pp 156–8, for the particular case $m = 1$.)

Consider the vector field on $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ $\frac{\mathrm{d}}{\mathrm{d}s}$, which is the horizontal (or total) lifting of $\frac{\partial}{\partial s}$ as a vector field on $(M, \mathcal{A})$. (See [Mon 92] for definitions. Here, it suffices to know its local expression $\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\partial}{\partial s} + x_s^a \frac{\partial}{\partial x^a} + x_{ts}^a \frac{\partial}{\partial x_t^a}$. See also remark 2 below.) Next, we take $\mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}$ and go over the graded bundle $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$. Define

$$\Theta^L := \mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}} \Theta_0^L$$

so that

$$\Theta^L = \left(\mathrm{d}^G x_s^a - \mathrm{d}^G t \cdot x_{ts}^a\right) \frac{\partial L}{\partial x_t^a} + (-1)^{|x^a|}\left(\mathrm{d}^G x^a - \mathrm{d}^G t \cdot x_t^a - \mathrm{d}^G s \cdot x_s^a\right) \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x_t^a} + \mathrm{d}^G t \cdot \frac{\mathrm{d}L}{\mathrm{d}s}.$$

Now, applying d^G to this expression, we will arrive at the Poincaré–Cartan 2-form on $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ (with the appropriate condition of regularity in L, namely $\det\left(\frac{\partial^2 L}{\partial x_t^a \partial x_t^b}\right)^\sim \neq 0$),

$$\Omega_L = \mathrm{d}^G \Theta^L.$$

It is easy to see that the local expression of Ω_L is

$$\begin{aligned}\Omega_L = {} & \mathrm{d}^G t\, \mathrm{d}^G x_{ts}^a \cdot \frac{\partial L}{\partial x_t^a} - \left(\mathrm{d}^G x_s^a - \mathrm{d}^G t \cdot x_{ts}^a\right) \mathrm{d}^G \left(\frac{\partial L}{\partial x_t^a}\right) \\ & + (-1)^{|x^a|}\left(\mathrm{d}^G t\, \mathrm{d}^G x_t^a - \mathrm{d}^G s\, \mathrm{d}^G x_s^a\right) \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x_t^a} \\ & - (-1)^{|x^a|}\left(\mathrm{d}^G x^a - \mathrm{d}^G t \cdot x_t^a - \mathrm{d}^G s \cdot x_s^a\right) \mathrm{d}^G \left(\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x_t^a}\right) - \mathrm{d}^G t\, \mathrm{d}^G \left(\frac{\mathrm{d}L}{\mathrm{d}s}\right). \end{aligned} \tag{3.2}$$

It is easy to check that this form is preserved by coordinate changes, so it is a well-defined global graded 2-form.

Remark 1. The coordinates on $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ are $\left\{t, s, x^a, x_t^a, x_s^a, x_{ts}^a, x_{tt}^a\right\}$, but we can see from (3.2) that Ω_L really depends only on $\left\{t, s, x^a, x_t^a, x_s^a, x_{ts}^a\right\}$. Thus, it is very interesting to be able to construct a space with these coordinates to project this 2-form. Such a space is an intermediate graded bundle between $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ and $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$, and has been constructed in [Mon-Muñ 02]. We shall not repeat its construction here, but refer the reader to that paper for details (see also the last section). We will denote this space by $J^{1|1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$.

Remark 2. Let us make some comments about the use of $\mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}$ which may seem unnatural at first sight. In a graded manifold $(M, \mathcal{A})$ with coordinates $\{x^i, x^{-j}\}_{1\leqslant i\leqslant m}^{1\leqslant j\leqslant n}$, we have two different notions of integral: the Berezin and the graded one. If we are given a superfunction $f = f_0 + f_j x^{-j} + \cdots + f_I x^{-I}$ (where $I = \{1, \ldots, n\}$), their respective actions are

$$\int_{\mathrm{Ber}} f = f_I \qquad \text{and} \qquad \int_{\mathrm{Grad}} f = f_0.$$

Accordingly, when dealing with the graded calculus of variations one introduces two different notions of Lagrangian densities: the Berezinian densities and the graded densities. For a graded submersion $p : (M, \mathcal{A}) \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$, the first order Berezinian Lagrangian densities have the form

$$\left[\mathrm{d}^G t \otimes \frac{\mathrm{d}}{\mathrm{d}s}\right] \cdot L \tag{3.3}$$

where $L \in \mathcal{A}^1_{J_G}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$, and the k-order graded Lagrangian densities are

$$\mathrm{d}^G t \cdot K$$

with $K \in \mathcal{A}^k_{J_G}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$. The crucial point is that to any first order Berezinian Lagrangian density as (3.3), we can associate a second order graded Lagrangian density

$$\mathrm{d}^G t \cdot \frac{\mathrm{d}L}{\mathrm{d}s} \tag{3.4}$$

in such a way that the solutions of the variational equations they induce are the same (Comparison Theorem, see [Mon-Mun 92]. Now, we observe that (3.4) can be written as $\mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}(\mathrm{d}^G t \cdot L)$, so the operator $\mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}$ passes from Berezinian densities to graded densities, though increasing the order. The advantage is that we can apply the well known graded techniques to problems with a Berezinian origin.

4. The graded Liouville vector field

In the classical setting for autonomous Lagrangians L, one constructs a symplectic form associated with L on TM, ω_L, and a function called the energy E_L, in such a way that the integral curves of the vector field ξ_L, given by $i_{\xi_L}\omega_L = \mathrm{d}E_L$, are the solutions to Euler–Lagrange equations. This is achieved with the introduction of a certain canonical vector field, the Liouville vector field Δ; the definition of the energy is then

$$E_L = L - \Delta L.$$

These notions are used as the base of the following development.

A graded vector field on the space of solutions will be called projectable if it has the form

$$X = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \tilde{X}$$

where $\tilde{X}$ is its vertical part. If X is such a field, then

$$\iota_X \Omega_L = \iota_{\frac{\partial}{\partial t}} \Omega_L + \iota_{\frac{\partial}{\partial s}} \Omega_L + \iota_{\tilde{X}} \Omega_L$$

and

$$\iota_X \Omega_L = 0 \qquad \text{if and only if} \qquad \iota_{\tilde{X}} \Omega_L = -\big(\iota_{\frac{\partial}{\partial t}} \Omega_L + \iota_{\frac{\partial}{\partial s}} \Omega_L\big).$$

To compute $\iota_{\frac{\partial}{\partial t}} \Omega_L + \iota_{\frac{\partial}{\partial s}} \Omega_L$ we only need to know the factors of $\mathrm{d}^G t, \mathrm{d}^G s$ in the local expression for Ω_L (3.2); these are, for $\mathrm{d}^G t$

$$\mathrm{d}^G x^a_{ts} \cdot \frac{\partial L}{\partial x^a_t} + x^a_{ts}\, \mathrm{d}^G \left(\frac{\partial L}{\partial x^a_t}\right) + (-1)^{|x^a|}\, \mathrm{d}^G x^a_t \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x^a_t} + (-1)^{|x^a|} x^a_t\, \mathrm{d}^G \left(\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x^a_t}\right) - \mathrm{d}^G \left(\frac{\mathrm{d}L}{\mathrm{d}s}\right)$$

and for $\mathrm{d}^G s$

$$(-1)^{|x^a|}\, \mathrm{d}^G x^a_s \left(\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x^a_t}\right) + (-1)^{|x^a|} x^a_s\, \mathrm{d}^G \left(\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x^a_t}\right).$$

Let us rewrite these factors in a more appropriate way. The factor of $\mathrm{d}^G t$ is

$$\begin{aligned}
\mathrm{d}^G &\left(x^a_{ts} \frac{\partial L}{\partial x^a_t} \right) + (-1)^{|x^a|}\, \mathrm{d}^G \left(x^a_t \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x^a_t} \right) - \mathrm{d}^G \left(\frac{\mathrm{d}L}{\mathrm{d}s} \right) \\
&= \mathrm{d}^G \left(x^a_{ts} \frac{\partial L}{\partial x^a_t} + (-1)^{|x^a|} x^a_t \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x^a_t} - \frac{\mathrm{d}L}{\mathrm{d}s} \right) \\
&= \mathrm{d}^G \mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}} \left(x^a_t \frac{\partial L}{\partial x^a_t} - L \right) = \mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}\, \mathrm{d}^G (\Delta_1 L - L).
\end{aligned}$$

On the other hand, the factor of $\mathrm{d}^G s$ can be written as

$$\mathrm{d}^G \left((-1)^{|x^a|} x^a_s \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x^a_t} \right) = \mathrm{d}^G \mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}} \left(-x^a_s \frac{\partial L}{\partial x^a_t} \right) = -\mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}\, \mathrm{d}^G (\Delta_2 L).$$

So, our condition for $\iota_X \Omega_L = 0$ now reads

$$\iota_{\tilde{X}} \Omega_L = -\mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}\, \mathrm{d}^G (\Delta_1 L - L) + \mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}}\, \mathrm{d}^G (\Delta_2 L) = \mathrm{d}^G \mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}} (L - \Delta L).$$

Definition 1. *We will call the graded Liouville vector field the field on* $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$*:*

$$\Delta := \Delta_1 - \Delta_2 = \left(x^a_t - x^a_s \right) \frac{\partial}{\partial x^a_t}. \tag{4.1}$$

Remark 3. The graded Liouville vector field can be constructed intrinsically (see [Car-Fig 97] for details). Note also that, being defined on $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$, Δ can be lifted to $J^{1|1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$.

Definition 2. *The energy associated with the Lagrangian L is*

$$E_L := \mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}} (L - \Delta L) = \mathcal{L}^G_{\frac{\mathrm{d}}{\mathrm{d}s}} \left(L - \left(x^a_t - x^a_s \right) \frac{\partial L}{\partial x^a_t} \right). \tag{4.2}$$

Then, we have

$$\iota_X \Omega_L = 0 \qquad \text{if and only if} \qquad \iota_{\tilde{X}} \Omega_L = \mathrm{d}^G E_L. \tag{4.3}$$

5. Graded semisprays and Euler–Lagrange equations for non-autonomous super-Lagrangians

Under the assumptions of regularity for L, a graded vector field $\tilde{X}^L$ solution of (4.3) determines a unique $X^L = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \tilde{X}^L$ (graded vector field on the space $J^{1|1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$) which is a solution of

$$\begin{cases} \iota_{X^L} \Omega_L = 0 \\ \iota_{X^L}\, \mathrm{d}^G t = 1 \\ \iota_{X^L}\, \mathrm{d}^G s = 1. \end{cases} \tag{5.1}$$

Definition 3. *This graded vector field X^L is called the (graded) Euler–Lagrange vector field.*

Remark 4. This setting resembles that of the classical cosymplectic manifolds (see [Lib 59, Alb 89, CLL 92]). A cosymplectic manifold is a triple (M, θ, ω) consisting of a smooth $(2n+1)$-dimensional manifold M with a closed 1-form θ and a closed 2-form ω, such that $\theta \wedge \omega^n \neq 0$. The standard example of a cosymplectic manifold is provided by an 'extended cotangent bundle' $(\mathbb{R} \times T^*N,\ \mathrm{d}t, \pi^*\, \mathrm{d}\lambda)$, with $t : \mathbb{R} \times T^*N \to \mathbb{R}$ and $\pi : \mathbb{R} \times T^*N \to T^*N$

the canonical projections and λ the canonical Liouville 1-form on T^*N. On a cosymplectic manifold M there exists a distinguished vector field R, the Reeb vector field, defined by

$$i_R\theta = 1 \qquad i_R\omega = 0.$$

From (5.1) we see that we are considering the graded analogue of the cosymplectic structure of classical mechanics, so X^L could also be called the graded Reeb vector field. However, we will follow more closely the equivalent approach of Saunders and Crampin (see [Sau 89]), based on the notion of semisprays. The reason is that this formalism extends inmediately to the case of fields in a covariant way, which is difficult in the cosymplectic framework.

Now, the idea is to develop conditions (5.1) to obtain the Euler–Lagrange superequations, but we will need to introduce some technical concepts first. For brevity, we will write $J^{1|1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ as $J_G^{1|1}(\pi)$.

Consider the 1-forms on $J_G^{1|1}(\pi)$ (called the contact 1-forms)

$$\Phi^a := \mathrm{d}^G x^a - \mathrm{d}^G t \cdot x_t^a - \mathrm{d}^G s \cdot x_s^a \qquad \Psi^a := \mathrm{d}^G x_s^a - \mathrm{d}^G t \cdot x_{ts}^a.$$

Let $c : \mathbb{R}^{1|1} \to (M, \mathcal{A})$ a graded curve; its $1|1$-jet prolongation is a section, denoted by $j^1 c$, of $J_G^{1|1}(\pi)$. A local section of $J_G^{1|1}(\pi)$, seen as a graded curve $\bar{c}$ on $J_G^{1|1}(\pi)$, is the $1|1$-jet prolongation of a graded curve on $(M, \mathcal{A})$ if and only if

$$\bar{c}^*\Phi^a = 0 \qquad \bar{c}^*\Psi^a = 0$$

this amounts to saying that, if $\left\{t, s, x^a, x_t^a, x_s^a, x_{ts}^a\right\}$ are the local coordinates of $J_G^{1|1}(\pi)$ then, along this section (to be closer to common practice in physics, in the following we omit the evaluation on the section):

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}s}\right) x^a = x_t^a + x_s^a \qquad \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}s}\right) x_s^a = x_{ts.}^a$$

Definition 4. *A graded semispray is a projectable vector field $X \in \mathcal{X}_G\left(J_G^{1|1}(\pi)\right)$ such that its integral curves are $1|1$-jet prolongations, i.e,*

$$\iota_X \, \mathrm{d}^G t = 1 = \mathrm{d}^G s \qquad \iota_X \Phi^a = 0 = \iota_X \Psi^a.$$

Thus, a graded semispray is given locally by

$$X = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \left(x_t^a + x_s^a\right)\frac{\partial}{\partial x^a} + A^a \frac{\partial}{\partial x_t^a} + x_{ts}^a \frac{\partial}{\partial x_s^a} + B^a \frac{\partial}{\partial x_{ts}^a}.$$

In the classical case, a semispray can be characterized through the vertical endomorphism $\tilde{J} = \frac{\partial}{\partial x_t^i} \otimes \mathrm{d}x^i$ and the Liouville vector field $\Delta = x_t^a \frac{\partial}{\partial x_t^a}$; thus, $X \in \mathcal{X}(J^1(\mathbb{R}, M))$ is a semispray if and only if $JX = 0$ and $\tilde{J}X = \Delta$, where $J = \tilde{J} - \Delta \otimes \mathrm{d}t$. The graded situation is somewhat different.

Proposition 1. *If $X \in \mathcal{X}_G\left(J_G^{1|1}(\pi)\right)$ is a graded semispray, then*

$$JX = 0 \qquad \textit{and} \qquad \tilde{J}X = \Delta.$$

Proof. It suffices to take into account the local expressions for X, J, $\tilde{J}$ and Δ. □

However, the converse is not true in general. Let us show a counterexample: an arbitrary $X \in \mathcal{X}_G\left(J_G^{1|1}(\pi)\right)$ has the local expression

$$X = f\frac{\partial}{\partial t} + g\frac{\partial}{\partial s} + A^a \frac{\partial}{\partial x^a} + B^a \frac{\partial}{\partial x_t^a} + C^a \frac{\partial}{\partial x_s^a} + D^a \frac{\partial}{\partial x_{ts}^a}$$

with f, g depending on the supercoordinates $\{t, s, x^a, x^a_t, x^a_s, x^a_{ts}\}$. If $JX = 0$ and $\tilde{J}X = \Delta$, all we can say is that $A^a = x^a_t + x^a_s$, and $fx^a_t + gx^a_s - x^a_t - x^a_s = 0$; but if we take $f = x^a_s + 1$, $g = 1 - x^a_t$, we have

$$x^a_s x^a_t + x^a_t + x^a_s - x^a_t x^a_s - x^a_t - x^a_s = 0$$

so $JX = 0$ and $\tilde{J}X = \Delta$, but X is not a semispray. Note that even when X is a projectable graded vector field, the reciprocal is not true, as $C^a \neq x^a_{ts}$ in general.

In order to make explicit computations, we will need the following result.

Proposition 2. *The local expression of the Poincaré–Cartan 2-form Ω_L, is*

$$\begin{aligned}
\Omega_L = {} & \mathrm{d}^G s\, \mathrm{d}^G s \cdot x^a_s \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial^2 L}{\partial s \partial x^a_t} \\
& + \mathrm{d}^G t\, \mathrm{d}^G s \left(-(-1)^{|x^a|} \frac{\mathrm{d}}{\mathrm{d}s} \left(x^a_t \frac{\partial^2 L}{\partial s \partial x^a_t} \right) + \frac{\mathrm{d}}{\mathrm{d}s} \left(x^a_s \frac{\partial^2 L}{\partial t \partial x^a_t} \right) + \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial s} \right) \\
& + (-1)^{|x^b|}\, \mathrm{d}^G t\, \mathrm{d}^G x^b \cdot \frac{\mathrm{d}}{\mathrm{d}s} \left(-\frac{\partial L}{\partial x^b} + (-1)^{|x^a||x^b|} x^a_t \frac{\partial^2 L}{\partial x^b \partial x^a_t} + \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial^2 L}{\partial t \partial x^b_t} \right) \\
& + (-1)^{|x^a_s||x^b|}\, \mathrm{d}^G t\, \mathrm{d}^G x^b_t \cdot \frac{\mathrm{d}}{\mathrm{d}s} \left(x^a_t \frac{\partial^2 L}{\partial x^b_t \partial x^a_t} \right) + \mathrm{d}^G t\, \mathrm{d}^G x^b_s \left(-\frac{\partial L}{\partial x^b} + \frac{\partial^2 L}{\partial t \partial x^b_t} \right. \\
& \left. + (-1)^{|x^a||x^b|} x^a_t \frac{\partial^2 L}{\partial x^b \partial x^a_t} - (-1)^{|x^b|} \frac{\mathrm{d}}{\mathrm{d}s} \left(-\frac{\partial L}{\partial x^b_s} + (-1)^{|x^a||x^b|} x^a_t \frac{\partial^2 L}{\partial x^b_s \partial x^a_t} \right) \right) \\
& + (-1)^{|x^a||x^b|}\, \mathrm{d}^G t\, \mathrm{d}^G x^b_{ts} \cdot x^a_t \frac{\partial^2 L}{\partial x^b_t \partial x^a_t} \\
& - \mathrm{d}^G s\, \mathrm{d}^G x^b \left(\frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial^2 L}{\partial s \partial x^b_t} + (-1)^{|x^a||x^b|} \frac{\mathrm{d}}{\mathrm{d}s} \left(x^a_s \frac{\partial^2 L}{\partial x^b \partial x^a_t} \right) \right) \\
& - (-1)^{|x^a||x^b|}\, \mathrm{d}^G s\, \mathrm{d}^G x^b_t \cdot \frac{\mathrm{d}}{\mathrm{d}s} \left(x^a_s \frac{\partial^2 L}{\partial x^b_t \partial x^a_t} \right) \\
& - \mathrm{d}^G s\, \mathrm{d}^G x^b_s \left((-1)^{|x^a||x^b_s|} \frac{\mathrm{d}}{\mathrm{d}s} \left(x^a_s \frac{\partial^2 L}{\partial x^b_s \partial x^a_t} \right) + (-1)^{|x^b_s|} \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial L}{\partial x^b_t} \right. \\
& \left. + (-1)^{|x^a_s||x^b|} x^a_s \frac{\partial^2 L}{\partial x^b \partial x^a_t} - (-1)^{|x^b|} \frac{\partial^2 L}{\partial s \partial x^b_t} \right) \\
& - (-1)^{|x^a_s||x^b|}\, \mathrm{d}^G s\, \mathrm{d}^G x^b_{ts} \cdot x^a_s \frac{\partial^2 L}{\partial x^b_t \partial x^a_t} - (-1)^{|x^a|+|x^b|}\, \mathrm{d}^G x^a\, \mathrm{d}^G x^b \cdot \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial^2 L}{\partial x^b \partial x^a_t} \\
& - (-1)^{|x^a|+|x^b|}\, \mathrm{d}^G x^a\, \mathrm{d}^G x^b_t \cdot \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial^2 L}{\partial x^b_t \partial x^a_t} \\
& - \mathrm{d}^G x^a\, \mathrm{d}^G x^b_s \left((-1)^{|x^a|+|x^b_s|} \frac{\mathrm{d}}{\mathrm{d}s} \frac{\partial^2 L}{\partial x^b_s \partial x^a_t} + (-1)^{|x^a|} \frac{\partial^2 L}{\partial x^b \partial x^a_t} \right) \\
& - \mathrm{d}^G x^a_s\, \mathrm{d}^G x^b \cdot \frac{\partial^2 L}{\partial x^b \partial x^a_t} - (-1)^{|x^a|}\, \mathrm{d}^G x^a\, \mathrm{d}^G x^b_{ts} \cdot \frac{\partial^2 L}{\partial x^b_t \partial x^a_t} \\
& - \mathrm{d}^G x^a_s\, \mathrm{d}^G x^b_t \cdot \frac{\partial^2 L}{\partial x^b_t \partial x^a_t} - \mathrm{d}^G x^a_s\, \mathrm{d}^G x^b_s \cdot \frac{\partial^2 L}{\partial x^b_s \partial x^a_t}.
\end{aligned}$$

Proof. A very long but standard computation. □

Now, consider the Euler–Lagrange graded vector field $X^L = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \tilde{X}^L$ which is a solution of (5.1).

Proposition 3. *The Euler–Lagrange field is a graded semispray.*

Proof. Consider the factors containing $\mathrm{d}^G x^b_{ts}$ and $\mathrm{d}^G x^b_t$ in Ω_L. From proposition 2, these are for $\mathrm{d}^G x^b_{ts}$

$$(-1)^{|x^a||x^b|}\,\mathrm{d}^G t\,\mathrm{d}^G x^b_{ts}\cdot x^a_t\frac{\partial^2 L}{\partial x^b_t\partial x^a_t} - (-1)^{|x^a_s||x^b|}\,\mathrm{d}^G s\,\mathrm{d}^G x^b_{ts}\cdot x^a_s\frac{\partial^2 L}{\partial x^b_t\partial x^a_t}$$
$$- (-1)^{|x^a|}\,\mathrm{d}^G x^a\,\mathrm{d}^G x^b_{ts}\cdot\frac{\partial^2 L}{\partial x^b_t\partial x^a_t}$$

thus, contracting with $X^L = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} + A^a\frac{\partial}{\partial x^a} + B^a\frac{\partial}{\partial x^a_t} + C^a\frac{\partial}{\partial x^a_s} + D^a\frac{\partial}{\partial x^a_{ts}}$ and equating to zero, we obtain

$$A^a = x^a_t + x^a_s.$$

To prove that $C^a = x^a_{ts}$, we isolate the factors containing $\mathrm{d}^G x^b_t$; the statement follows then from the local characterization of semisprays. □

Definition 5. *If $X \in \mathcal{X}_G\big(J^{1|1}_G(\pi)\big)$ is a graded semispray, a graded curve $c : \mathbb{R}^{1|1} \to (M, \mathcal{A})$ is called a trajectory of X if its $1|1$-jet prolongation is an integral curve of X.*

Let us recall that, in the jet bundle $J^\infty_G(\pi)$

$$\frac{\mathrm{d}}{\mathrm{d}t}x^a_{ts} = x^a_{tts} \qquad \frac{\mathrm{d}}{\mathrm{d}s}x^a_t = x^a_{tt}$$

then locally c is a trajectory of X if and only if the following system of second-order differential equations is satisfied along the section induced by c:

$$\begin{cases} A^a = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}s}\right)x^a_t = x^a_{tt} + x^a_{ts} \\ B^a = \left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}s}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}s}\right)x^a_s = x^a_{tts}\end{cases}$$

(it is really a mixture of first- and second-order equations).

Theorem 1. *Let L be a regular Lagrangian on $J^{1|1}_G(\pi)$, and let X^L be the Euler–Lagrange semispray. Then the trajectories of X^L are the solutions of the Euler–Lagrange superequations*

$$\frac{\partial L}{\partial x^a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x^a_t} - (-1)^{|x^a|}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial x^b_s} = 0.$$

Proof. The technique is the same as in proposition 3: consider the factors containing $\mathrm{d}^G x^b_s$ in the equation $\iota_{X^L}\Omega_L = 0$ $\big($with $X^L = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} + \left(x^a_t + x^a_s\right)\frac{\partial}{\partial x^a} + B^a\frac{\partial}{\partial x^a_t} + x^a_{ts}\frac{\partial}{\partial x^a_s} + D^a\frac{\partial}{\partial x^a_{ts}}\big)$. □

6. The graded Noether theorem and supersymmetries

Our purpose in this section is to give a version of Noether's theorem on the existence of conserved quantities along the trajectories of the system valid in the graded case. There are a lot of classical formulations of this result, but not all of them can be used as a guide for the graded case. What we will present here is a graded version of the theorem directly related to the symplectic structure of the Lagrangian formalism, as it provides an interpretation of the energy as a conserved quantity related to invariance under 'supertime' translations and a connection with the t- and s-Hamiltonians of [Mon-Muñ 02].

The main ingredient is a set of transformations of the base manifold

$$\Phi : \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to (M, \mathcal{A})$$

forming a (1, 1)-supergroup (see [Mon-Sán 93] for details). This implies the existence of a graded derivation of $\mathcal{A}$, a supervector field T which can be lifted to a supervector field $\bar{T}$ on $J_G^{1|1}(\pi)$. Now, we state a property of these liftings.

Lemma 1. *The condition*

$$j^2(c)^*(\iota_X\Omega_L) = 0$$

for all X projectable supervector fields on the space $J_G^{1|1}(\pi)$*, is equivalent to the section c being a solution to the Euler–Lagrange superequations*

$$\frac{\partial L}{\partial x^a} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial x_t^a} - (-1)^{|x^a|}\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial x_s^a} = 0.$$

Proof. See theorem 8.1 in [Mon-Muñ 02]. □

This result reveals that, along sections that represent trajectories of the system, $\iota_{\bar{T}}\Omega_L = 0$. That is what we need to obtain conserved currents.

Definition 6. *A supervector field* $X \in X_G\big(J_G^{1|1}(\pi)\big)$ *is a supersymmetry of the Lagrangian problem associated with L if*

$$\mathcal{L}_X^G\Theta^L = 0. \tag{6.1}$$

Theorem 2. *If* $\bar{T} \in X_G\big(J_G^{1|1}(\pi)\big)$ *is a supersymmetry of the Lagrangian problem associated with L, then* $\iota_{\bar{T}}\Theta^L$ *is constant along the trajectories of the system. This quantity is called the Noether supercurrent associated with* $\bar{T}$.

Proof. Let us develop the condition of (6.1). We have

$$\mathcal{L}_{\bar{T}}^G\Theta^L = \iota_{\bar{T}}\,\mathrm{d}^G\Theta^L + \mathrm{d}^G\iota_{\bar{T}}\Theta^L = 0$$

and applying the preceding lemma, along the trajectories

$$\mathrm{d}^G\iota_{\bar{T}}\Theta^L = 0.$$ □

Now, let us recall that, in the classical setting, the energy is conserved whenever the system presents invariance under time translations. As an application of Noether's theorem, we will now deduce the expression of the superenergy under the assumption of non-explicit dependence on s, t in the super-Lagrangian; following our guiding principle (mentioned in the introduction), we consider the 'supertime' translations, induced by the supervector field

$$T = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}.$$

It is a straightforward computation to see that, when L does not depend explicitly on s, t, $\mathcal{L}_T^G\Theta^L = 0$. Then, we have the conserved current (see (4.2))

$$\begin{aligned}\iota_{\frac{\partial}{\partial s}+\frac{\partial}{\partial t}}\Theta^L &= -x_{ts}^a\frac{\partial L}{\partial x_t^a} - (-1)^{|x^a|}x_t^a\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial x_t^a} + \frac{\mathrm{d}L}{\mathrm{d}s} - (-1)^{|x^a|}x_s^a\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial L}{\partial x_t^a}\\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left(-\big(x_t^a - x_s^a\big)\frac{\partial L}{\partial x_t^a} + L\right) = E_L.\end{aligned}$$

Let us make the important remark that

$$E_L = H_t + H_s$$

where H_t and H_s are, respectively, the t-Hamiltonian and s-Hamiltonian introduced in [Mon-Muñ 02]:

$$H_t = -\frac{\mathrm{d}}{\mathrm{d}s}\left(x_t^a \frac{\partial L}{\partial x_t^a} - L\right) \qquad H_t = \frac{\mathrm{d}}{\mathrm{d}s}\left(x_s^a \frac{\partial L}{\partial x_t^a}\right).$$

We see here that these superfunctions, which appear as separate entities in the paper cited, have a natural interpretation taken together. Thus, we can recover the Hamiltonian formulation of supermechanics from within the Lagrangian one in the autonomous case, by defining the Hamiltonian as the Noether supercurrent associated with ‘ supertime’ translations (the superenergy), just as in the classical setting.

7. Super-Lagrangian formalism and Batalin–Vilkoviski master equation

Let us recall the basics of the Batalin–Vilkoviski formalism (see [Bat-Vil 81, Kos 95, Wit 90]). The theory deals with a set of fields Φ^A (A is a certain set of indices) and the corresponding antifields Φ^*_A, with parity reversed, that is, $|\Phi^*_A| = |\Phi^A| + 1$. Then, it searches for an action $W(\Phi, \Phi^*)$ satisfying some technical conditions that make it suitable to construct a quantum field theory; one of these is the invariance of a functional integral, which could be written as

$$Z = \int \mathrm{e}^{\frac{\mathrm{i}}{\hbar}W} \prod_A \mathrm{d}\Phi^A$$

under the so-called BRST transformations. This condition is equivalent to the famous master equation

$$\tfrac{1}{2}(W, W) = \mathrm{i}\hbar \Delta W \tag{7.1}$$

where the odd bracket (., .) is defined by

$$(F, H) = \frac{\partial_j F}{\partial \Phi^A}\frac{\partial_k H}{\partial \Phi^*_A} - \frac{\partial_j F}{\partial \Phi^*_A}\frac{\partial_k H}{\partial \Phi^A} \tag{7.2}$$

and Δ is the Batalin–Vilkoviski operator

$$\Delta = \frac{\partial_j}{\partial \Phi^A}\frac{\partial_k}{\partial \Phi^*_A}. \tag{7.3}$$

Note the important property of nilpotency:

$$\Delta^2 = 0.$$

Witten showed ([Wit 90], and see also [Sch 93]) that these formulae have an algebraic background: they make sense each time one has a Gerstenhaber algebra with a generating operator of square zero. Let us be more explicit.

Definition 7. *Let G be an associative, graded commutative algebra over a commutative field of characteristic 0. Let* $[\![., .]\!]$ *be an odd Poisson bracket on G with degree* -1. *The pair* $(G; [\![., .]\!])$ *is called a Gerstenhaber algebra structure on G.*

Remark 5. For generalities about graded Poisson brackets, see [Bel-Mon 95] or [Kos 95].

Definition 8. *A linear operator of degree* -1, $\Delta : G \to G$, *generates the graded bracket* $[\![., .]\!]$ *on G if*

$$[\![S, T]\!] = (-1)^{|S|}(\Delta(ST) - \Delta(S)T - (-1)^{|S|}S\Delta(T))$$

for all $S, T \in G$. *The Gerstenhaber algebra* $(G; [\![., .]\!])$ *is called a Batalin–Vilkoviski algebra (or an exact Gerstenhaber algebra) if* $\Delta^2 = 0$.

Remark 6. It is important to make the following observations about how to construct generating operators (see [Khu-Ner 93, Kos-Mon 01]). On a supermanifold $(M, \mathcal{A})$ with a Berezinian volume ξ, there is a divergence operator div^ξ, that maps derivations of $\mathcal{A}$ to $\mathcal{A}$. Given an odd Poisson bracket on A, the divergence of a Hamiltonian derivation up to a sign and a factor of $\frac{1}{2}$, is a generating operator of the odd Poisson bracket. More precisely, the operator $f \mapsto (-1)^{|f|}\frac{1}{2}\,\mathrm{div}^\xi [\![f, .]\!]$ $(f \in \mathcal{A})$ generates the odd Poisson bracket $[\![., .]\!]$.

Of course, the bracket (7.2) and the Batalin–Vilkoviski operator (7.3), give an example of Batalin–Vilkoviski algebra.

If $(G; [\![., .]\!])$ is a Batalin–Vilkoviski algebra, we say that $S \in G$ is a solution of the associated master equation if it verifies the analogue of (7.1):

$$\frac{1}{2}[\![S, S]\!] = \Delta S. \tag{7.4}$$

Our aim in the rest of this section is to prove the following result.

Theorem 3. *The space of solutions of the variational problem determined by a classical Lagrangian $L \in C^\infty(J^1(\pi : M \times \mathbb{R} \to \mathbb{R}))$, has the structure of a Batalin–Vilkoviski algebra, and the superenergy $S = E_L$ is a solution of the associated master equation.*

To achieve that, we will work with a specific supermanifold: that with its structural sheaf determined by its ring of differentiable functions, $(M, C^\infty(M))$; moreover, we will restrict ourselves to the autonomous case, that is, when there is no explicit dependence on s, t in the Lagrangian, so we will drop the factor $\mathbb{R}^{1|1}$. The steps to follow are: first, to construct out of the space $J^{1|1}_G(\pi)$ another one that can be identified with a symplectic supermanifold having a very simple structure (one of Koszul–Schouten type); this is precisely the space of solutions that appears in the statement of the theorem. Second, to construct a Batalin–Vilkoviski system, that is, a system whose evolution is determined by an odd graded Poisson bracket and a Hamiltonian function that verifies the master equation. The first step has already been considered in [Mon-Muñ 02], and here we will just review it briefly.

Proof. Consider $(M, C^\infty(M))$. In this supermanifold, there are no negative index supercoordinates so we will denote them by $\{x^i\}_{i=1}^{n=\dim M}$. A classical regular Lagrangian $L \in C^\infty(J^1(\pi : \mathbb{R} \times M \to M))$ can be lifted to $J^1(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ and then we can apply all the results of the previous sections. In particular, it can be seen that the space $J^{1|1}_G(\pi)$ projects onto another space, denoted by $(\mathcal{S}, \mathcal{A}_S)$ and which will be called the space of solutions, in which Ω_L is a symplectic form such that this space is graded isomorphic to $(TM, \Omega(TM))$ endowed with the Koszul–Schouten form Ξ_{KS} (theorem 14.5 in [Mon-Muñ 02]). If in TM we take the classical canonical coordinates given by L, $\left\{x^i, p^i = \frac{\partial L}{\partial x^i}\right\}_{i=1}^n$, then $\{x^i, p^i, x^{-i}, p^{-i}\}_{i=1}^n$ is a supercoordinate system on $(TM, \Omega(TM))$ and the Koszul–Schouten form has the aspect

$$\Xi_{\mathrm{KS}} = \mathrm{d}^G x^{-i}\,\mathrm{d}^G p^i + \mathrm{d}^G x^i\,\mathrm{d}^G p^{-i}.$$

Moreover, the super-Hamiltonian vector field corresponding to the superfunction $E_L (= H_t + H_s)$ is

$$X_{E_L} = -\mathcal{L}_{X_H} - \mathrm{d} \tag{7.5}$$

where H is the classical Hamiltonian associated with L, i.e, $H = x^i_t p^i - L$.

Let us note that the supersymplectic form Ω_L has parity $|L| + 1$, so if we want to obtain an associated odd Poisson bracket, we must take L as a homogeneous even Lagrangian.

It is well known (see [Bel-Mon 95]) that the Koszul–Schouten form Ξ_{KS}, has an associated graded Poisson bracket $[\![., .]\!]$ which is a Koszul–Schouten bracket generated by an operator of the type $\mathcal{L}_P$, where P is a Poisson bivector. In this case, $P = P_L$ is the Poisson bivector induced on TM by the Lagrangian L, and it is non-degenerate if L is regular (as we are supposing).

To summarize, what we have obtained from the homogeneous even super-Lagrangian L on $J_G^{1|1}(\pi)$, is an identification of the symplectic superspace $(\mathcal{S}, \mathcal{A}_S)$ endowed with Ω_L, with the symplectic supermanifold $(TM, \Omega(TM))$ equipped with the Koszul–Schouten form Ξ_L or, equivalently, with the odd Poisson bracket $[\![.,.]\!]$ generated by $\mathcal{L}_{P_L}$, where P_L is the Poisson bivector on TM induced by L. Note that, as a consequence of P_L being Poisson,

$$\mathcal{L}_{P_L}^2 = \tfrac{1}{2}\left[\mathcal{L}_{P_L}, \mathcal{L}_{P_L}\right] = \tfrac{1}{2}\mathcal{L}_{[P_L,P_L]_{SN}} = 0$$

so that the generating operator of the bracket is nilpotent.

Thus $((TM, \Omega(TM)); [\![.,.]\!])$ endowed with $\Delta = \mathcal{L}_{P_L}$ is a Batalin–Vilkoviski algebra. Now we turn our attention to the master equation.

As we are dealing with a super-Lagrangian L which is not explicitly dependent on t, s, we have a Hamiltonian to be identified with the superenergy (see section 5), let us call it S. In the coordinates we are working with (see [Mon-Muñ 02]), this is

$$S = E_L = H_t + H_s = \mathrm{d}H - \omega$$

where $\omega = \mathrm{d}x^i\,\mathrm{d}p^i$ is the symplectic form on TM induced by L as a classical Lagrangian.

The 'supertime evolution' of an observable (superfunction) $\alpha \in \Omega(TM)$, is defined through

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)\alpha = [\![S, \alpha]\!].$$

Of course, the Hamiltonian is a constant of motion. We know it from the graded Noether theorem of section 6, but now we can see it explicitly: as the degree of $[\![.,.]\!]$ is -1 and those of the s- and t- Hamiltonians are 0 and 1, respectively, we have

$$[\![S, S]\!] = [\![H_t + H_s, H_t + H_s]\!] = [\![H_s, H_s]\!]$$

but to H_s corresponds to the Hamiltonian supervector field $X_{H_s} = \mathrm{d}$ (the exterior differential on TM), see [Bel-Mon 95], so we get the classical Batalin–Vilkoviski master equation:

$$[\![S, S]\!] = [\mathrm{d}, \mathrm{d}] = 0 \tag{7.6}$$

(here, $[.,.]$ is the graded bracket on endomorphisms of $\Omega(TM)$).

Now, applying the graded Jacobi identity for the odd Poisson bracket $[\![.,.]\!]$, we get for any $\alpha \in \Omega(TM)$

$$[\![S, [\![S, \alpha]\!]]\!] = 0.$$

This result states that the so-called classical BRST operator, $[\![S, .]\!] = X_S = -\mathcal{L}_{X_H} - \mathrm{d}$ (recall (7.5)) is also nilpotent.

By virtue of (7.6), S will be a solution of the master equation (7.4) if $\Delta S = 0$. But this is very easy to see (recall remark 3): $\Delta = \mathcal{L}_{P_L}$ as a generating operator is the divergence of the Hamiltonian derivation with respect to the canonical Berezinian ξ on the supermanifold $(TM, \Omega(TM))$ (see [Kos-Mon 01] for the construction of ξ), so

$$\Delta S = (-1)^{|S|}\tfrac{1}{2}\,\mathrm{div}^{\xi}[\![S, .]\!] = \tfrac{1}{2}\,\mathrm{div}^{\xi} X_{H_s} - \tfrac{1}{2}\,\mathrm{div}^{\xi} X_{H_t} = -\tfrac{1}{2}\,\mathrm{div}^{\xi}\,\mathrm{d} + \tfrac{1}{2}\,\mathrm{div}^{\xi}\mathcal{L}_{X_H}$$

but in [Kos-Mon 01] it is shown that $\mathrm{div}^{\xi}\,\mathrm{d} = 0 = \mathrm{div}^{\xi}\mathcal{L}_{X_H}$, so we have

$$\Delta S = 0$$

and, consequently, S is a solution of the master equation. □

Remark 7. Let us note that the generating operator $\Delta = \mathcal{L}_{P_L}$, when written in coordinates, takes the form of the original Batalin–Vilkoviski operator (7.3).

Corollary 1. *The superfunction* $\exp(S)$ *is also a solution of the Batalin–Vilkoviski master equation.*

Proof. Recall that, from the definition of the generating operator,

$$\mathcal{L}_{P_L}(\alpha\beta) = (-1)^{|\alpha|}\big([\![\alpha, \beta]\!] - \mathcal{L}_{P_L}(\alpha)\beta - (-1)^{|\alpha|}\alpha\mathcal{L}_{P_L}(\beta)\big) \qquad \forall\alpha, \beta \in \Omega(M).$$

Then,

$$\mathcal{L}_{P_L}(\mathrm{e}^S) = \mathcal{L}_{P_L}(\mathrm{e}^{-\omega}\,\mathrm{e}^{\mathrm{d}H}) = [\![\mathrm{e}^{-\omega},\ \mathrm{e}^{\mathrm{d}H}]\!] - \mathcal{L}_{P_L}(\mathrm{e}^{-\omega})\,\mathrm{e}^{\mathrm{d}H} - \mathrm{e}^{-\omega}\mathcal{L}_{P_L}(\mathrm{e}^{\mathrm{d}H})$$

but we have $\mathrm{e}^{\mathrm{d}H} = 1 + \mathrm{d}H$, so $\mathcal{L}_{P_L}\,\mathrm{e}^{\mathrm{d}H} = 0$. On the other hand, $\mathcal{L}_{P_L}\omega = 0$, and this implies $\mathcal{L}_{P_L}\,\mathrm{e}^{-\omega} = 0$ as well. Thus, only the term $[\![\mathrm{e}^{-\omega},\ \mathrm{d}H]\!]$ remains to be calculated; but from the definition of the Koszul–Schouten bracket, $[\![\mathrm{e}^{-\omega},\ \mathrm{d}H]\!] = -\mathcal{L}_{X_H}\,\mathrm{e}^{-\omega} = 0$. What we have obtained is

$$\mathcal{L}_{P_L}(\mathrm{e}^S) = 0.$$

Moreover, $[\![\mathrm{e}^S,\ \mathrm{e}^S]\!]$ gives us three terms, and all of them vanish: applying Leibniz's rule, $[\![\mathrm{e}^{-\omega},\ \mathrm{e}^{-\omega}]\!]$ reduces to terms of the form $[\![\omega,\ \mathrm{e}^{-\omega}]\!] = \mathrm{d}\,\mathrm{e}^{-\omega} = 0$; on the other hand, the other two terms contain $\mathrm{e}^{\mathrm{d}H} = 1 + \mathrm{d}H$, so

$$[\![\mathrm{e}^{-\omega},\ \mathrm{e}^{\mathrm{d}H}]\!] = [\![\mathrm{e}^{-\omega},\ \mathrm{d}H]\!] = 0 \qquad [\![\mathrm{e}^{\mathrm{d}H},\ \mathrm{e}^{\mathrm{d}H}]\!] = [\![\mathrm{d}H,\ \mathrm{d}H]\!] = 0$$

that is, $[\![\mathrm{e}^S,\ \mathrm{e}^S]\!] = 0$ and

$$\Delta\,\mathrm{e}^S = 0 = [\![\mathrm{e}^S,\ \mathrm{e}^S]\!].$$

□

Remark 8. Note that in the theorem and the corollary, we have obtained solutions to (7.4) in a very strong sense, as both members of the equation vanish separately.

We must insist that we have obtained these results within the super-Lagrangian formalism, but working with the ‘supertime’ manifold $\mathbb{R}^{1|1}$ instead of the usual $\mathbb{R}$ (see [Ibo-Mar 93] or [Car-Fig 97] for treatments based on $\mathbb{R}$, and note the differences in the Poincaré–Cartan form Ω_L with respect to ours). There have also been formulations with an ‘odd time’ τ, which would correspond to $\mathbb{R}^{0|1}$: one such can be consulted in [Day 88]; in that work, some questions about the relation between L and S are raised. We hope we have answered them, offering a framework exhibiting all the features of the classical one. In particular, the solution obtained to the Batalin–Vilkoviski master equation, S, can be viewed as the superenergy associated with L through the super-Lagrangian formalism.

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